# INVARIANT MEASURES FOR NON-REGULAR RANDOM DYNAMICAL SYSTEMS [1]

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The classical notion of a random dynamical system has proved to be very fruitful in many applications to stochastic differential equations (see, e.g. [1] and references therein). Roughly speaking, a random dynamical system (RDS), or more exactly, a *regular* random dynamical system, is a combination of a measure preserving flow on a given probability space with a smooth (or just continuous) dynamical system, typically generated by a differential or difference equation. However, there are some natural examples of differential equations, especially in infinite dimensional spaces, which do not produce smooth or even continuous dynamical systems and cannot therefore be regarded as RDS in the above classical sense. The most prominent example here is given by stochastic delay equations with a delayed diffusion term (see e.g. [2]). Such equations (and many others) give rise to *non-regular random dynamical systems*.

In this contribution we discuss a notion of a generalized RDS, which covers both regular RDS in the sense of [1] and non-regular RDS coming from stochastic delay differential equations and stochastic parabolic differential equations. The basic idea is as follows. The core of the definition of the classical smooth (or continuous) RDS is constituted by two objects: the shift operator generating the measure preserving dynamical system and a Carathéodory function, which provides an analogue of the smooth deterministic dynamical system in the stochastic case. The classical cocycle condition is then imposed on the composition of the shift with the corresponding Nemystkiĭ operator generated by this Carathéodory function [1]. In our generalization the Nemystkiĭ operators generated by Carathéodory functions are replaced by the so-called *local operators*, the shifts on the underlying probability space being the same as before. In other words, the conventional RDS is a nonlinear shift operator, while the generalized RDS is a so-called *atomic operator* introduced in [3]. Such operators amount to compositions of local operators with shifts and in general cannot be reduced to conventional nonlinear shift operators.

Introducing *the generalized cocycle property* we show that this approach is well adapted to extend the classical notion of RDS to a much wider class of more general stochastic equations including stochastic functional differential equations and equations in infinite-dimensional spaces. We then extend the fundamental result on existence of invariant measures for RDS (Theorem 1.5.10 in [1]) to generalized RDS.

Let $(\Omega, \Sigma, P)$ be a complete probability space. Let us also assume $X$ to be a separable metrizable space, and $B(X)$ to be its Borel $\sigma$-algebra; $C^b(X)$ to stand for the space of all real-valued continuous bounded functions on $X$ equipped with the supremum norm $|| \cdot ||_\infty$; $Car^0(\Omega, \Sigma, P; X)$ to denote the set of real-valued Carathéodory functions satisfying $f(\omega, \cdot) \in C^b(X)$ and

$$\int_\Omega ||f(\omega, \cdot)||_\infty \, dP(\omega) < +\infty.$$

The set $\mathcal{Y}(\Omega, \Sigma, P; X)$ of Young measures consists of all positive measures $\nu$ on $\Omega \times X$, whose projections onto $\Omega$ (i.e. the image measures $\pi_{\Omega\#}\nu$ under the projection map $\pi_\Omega \colon \Omega \times X \to \Omega$

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defined by $\pi_\Omega(\omega, x) := \omega$) equal $P$, i.e. $\nu(A \times X) = P(A)$ for each $A \in \Sigma$. The elements of $\mathcal{Y}(\Omega, \Sigma, P; X)$ are called Young measures with marginal $P$.

The set of Young measures $\mathcal{Y}(\Omega, \Sigma, P; X)$ is supposed to be endowed with the narrow topology, i.e. the weakest topology which makes all the maps

$$\nu \in \mathcal{Y}(\Omega, \Sigma, P; X) \mapsto \int_{\Omega\times X} f\, d\nu$$

continuous, where $f \in Car^0(\Omega, \Sigma, P; X)$.

T h e o r e m 1. *Every nonlinear continuous atomic operator* $T$*:* $L^0(\Omega_1, \Sigma_1, P_1; X_1) \to$ $\to L^0(\Omega_2, \Sigma_2, P_2; X_2)$ *admits a continuous* (*in the narrow topology*) *extension*

$$\bar{T}:\ \mathcal{Y}(\Omega_1, \Sigma_1, P_1; X_1) \to \mathcal{Y}(\Omega_2, \Sigma_2, P_2; X_2).$$

An atomic operator is here a composition $T = N \circ T_F$ of a continuous local operator $N : L^0(\Omega_2, F\Sigma_1, P_2; X_1) \to L^0(\Omega_2, \Sigma_2, P_2; X_2)$ and a generalized shift operator $T_F : L^0(\Omega_1, \Sigma_1, P_1; X_1) \to L^0(\Omega_2, F\Sigma_1, P_2; X_1)$ generated by a measure preserving mapping $F : \Sigma_1 \to \Sigma_2$.

D e f i n i t i o n 1. A set of Young measures $\mathcal{H} \in \mathcal{Y}(\Omega, \Sigma, P; X)$ is called *tight*, if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset X$ such that

$$\sup_{\nu\in\mathcal{H}} \nu(\Omega \times (X \setminus K_\varepsilon)) \leqslant \varepsilon.$$

Likewise, a set of functions $\mathcal{K} \subset L^0(\Omega, \Sigma, P; X)$ is called tight, if it is tight as a set of Young measures, i.e. the set $\{\delta_u\}_{u\in\mathcal{K}}$ is tight. In other words, $\mathcal{K} \subset L^0(\Omega, \Sigma, P; X)$ is tight, if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset X$ such that

$$\sup_{u\in\mathcal{K}} P(\{\omega \in \Omega\, :\, u(\omega) \notin K_\varepsilon\}) \leqslant \varepsilon.$$

Let $G$ be an additive subset of the set $R$ of all real numbers. Typical examples are $R^+$, $\gamma Z^+$ ($\gamma > 0$) etc.

As an immediate corollary of Theorem 1 we obtain the following result.

T h e o r e m 2. *Let* $\mathcal{K} \subset L^0(\Omega, \Sigma, P; X)$ *be a tight set. Let* $T_\tau$*:* $L^0(\Omega, \Sigma, P; X) \to$ $\to L^0(\Omega, \Sigma, P; X)$*, where* $\tau \in G$ *be a one parameter family of commuting continuous* (*in measure*) *atomic operators sending* $\mathcal{K}$ *into itself. Then this family admits a common invariant measure* $\nu \in \bar{\mathcal{K}}$*. In particular, every continuous* (*in measure*) *atomic operator* $T$*:* $L^0(\Omega, \Sigma, P; X) \to$ $\to L^0(\Omega, \Sigma, P; X)$ *sending* $\mathcal{K}$ *into itself admits a* (*generalized*) *invariant measure.*

## REFERENCES


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